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SHOCK-EXCITED VIBRATIONS OF A CONSERVATIVE DUFFING OSCILLATOR WITH APPLICATION TO SHOCK PROTECTION IN PORTABLE ELECTRONICS

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Abstract—Shock-excited vibrations of a conservative Duffing oscillator are examined in application to shock protection of a vulnerable product during accidental drop. We show that a hard characteristic of the oscillator might be advisable for a product which is able to withstand high accelerations (decelerations), while the maximum displacement has to be made small by any means. On the other hand, application of a spring with a soft characteristic can result in appreciably lower maximum accelerations (decelerations) than in a linear system and therefore can be recommended in the case when the requirement for the lowest displacement possible is not very stringent. However, if the drop height is not known with certainty (which is typically the case) the advantages of a soft spring cannot be utilized to the full extent, because of the possibility of a "rigid impact". In such a case a probabilistic approach can be effectively used to design a soft spring with a low enough probability of a rigid impact. The obtained results can be helpful, particularly, when designing spring protectors in portable electronics. The author believes that these results can be useful for a rather broad class of nonlinear springs, not necessarily with cubic restoring forces.

INTRODUCTION

Dynamic loading often plays a critical role in the functional performance and mechanical reliability of electronic components and devices (see e.g. Steinberg (1988), Suhir and Lee (1990), Engel (1993), Suhir (1992, 1993), Suhir and Burke (1994)). Such loading can be caused by accidental mishandling or misuse of the equipment in service, or can occur during its manufacturing, testing, or shipment (transportation). In military applications, dynamic loading occurs even during normal operation of the electronic equipment (Military Handbook, 1987).

The ability to predict and, possibly, to minimize the adverse consequences of dynamic loading on electronic equipment and devices is especially important for portable products. This is due not only to the fact that these products could be easily dropped, but, more importantly, because the maximum displacement ("stopping distance") of vulnerable structural elements employed in portable electronic products, when subjected to shock loads, has to be made very short. Therefore such elements can experience very high accelerations and, as a consequence of that, elevated dynamic stresses.

It has been shown (Suhir and Burke, 1994) that if a linear shock protector (spring) is used, the product of the maximum displacement, x_{max}, and the maximum acceleration (deceleration), $\ddot{x}_{max}$ (this product can be used as a suitable characteristic of the "quality" of a shock protector), is equal to

$$R = x_{max}\ddot{x}_{max} = -2gH, \tag{1}$$

where g is the acceleration due to gravity, and H is the drop height. Neither the spring constant K of the protector, nor the mass M of the element enter this formula. Hence, as long as a linear spring is used, very little can be done to optimize the dynamic response of the system.

Note that the relationship (1) simply follows from the equation of motion

$$M\ddot{x}+Kx = 0, \tag{2}$$

and the equation

$$MgH = \tfrac{1}{2}Kx_{max}^2 \tag{3}$$

of the energy balance at the moment of time when the initial potential energy (left part of this equation) is completely exhausted and is transferred into the strain energy (right part of the equation). Indeed, from (2) we have:

$$\ddot{x}_{max} = -\omega_0^2 x_{max}, \tag{4}$$

where

$$\omega_0 = \sqrt{\frac{K}{M}} \tag{5}$$

is the vibration frequency. Then the formula (1) can be obtained by writing the eqn (3), with consideration of (5), as

$$x_{max} = \frac{\sqrt{2gH}}{\omega_0}, \tag{6}$$

and eliminating the frequency ω_0 from the eqns (4) and (6). The eqn (2) reflects an obvious assumption that the drop height H is substantially larger than the value

$$H_{min} = \frac{g}{\omega_0^2} \tag{7}$$

and therefore the induced acceleration (deceleration) is significantly larger than g. This is usually the case in actual situations (Suhir and Burke (1994)).

The formula for the maximum linear acceleration (deceleration), $\ddot{x}_{max}$, can be obtained from (4) and (6) as

$$\ddot{x}_{max} = -\omega_0\sqrt{2gH}. \tag{8}$$

Note that the formulas (6) and (8) for the maximum displacement and the maximum acceleration were obtained without solving the eqn (2), i.e. without obtaining the expressions for the displacement and the acceleration for each moment of time.

The primary objective of the analysis which follows is to find out whether application of a nonlinear spring can lead to an appreciable reduction in the product R of the maximum displacement and the maximum acceleration. The analysis is limited, for the sake of simplicity, to the case of a Duffing oscillator (see, e.g., Stoker (1950))

$$\ddot{x}+\omega_0^2 x+\alpha x^3 = 0, \tag{9}$$

where the parameter α of the nonlinearity can be either positive (hard characteristic) or negative (soft characteristic). The initial conditions are:

$$x(0) = 0, \quad \dot{x}(0) = \sqrt{2gH}. \tag{10}$$

It is believed that the trends founds in this analysis hold, at least qualitatively, for a rather broad class of nonlinear springs, not necessarily with cubic restoring forces.

ANALYSIS

Hard spring

The solution to the eqn (9) can be sought, using elliptic functions (Suhir (1992,a,b)), in the form:

$$x = \frac{\sqrt{2gH}}{\omega_0}\eta_1 cn(\sigma t+\varepsilon, k) = \frac{\sqrt{2gH}}{\omega_0}\eta_1 cnu, \tag{11}$$

where cnu is the elliptic cosine (see, e.g. Bateman and Erdelyi (1995)), k is the modulus of the elliptic function, ε is the initial phase angle, σ is the frequency parameter, and η_1 is the factor which considers the effect of the nonlinearity on the amplitude of vibrations (maximum displacement). From (11), using the rules of the differentiation of the elliptic functions, we obtain:

$$\dot{x} = -\frac{\sqrt{2gH}}{\omega_0}\eta_1 \sigma snu\, dnu, \tag{12}$$

$$\ddot{x} = -\frac{\sqrt{2gH}}{\omega_0}\eta_1 \sigma^2 cnu(1-2k^2 sn^2 u), \tag{13}$$

where snu is the elliptic sine, and $dnu = \sqrt{1-k^2 sn^2 u}$ is the function of delta-amplitude.

Substituting (11) and (13) into the eqn (9), and using the initial conditions (10), we obtain the following relationships for the parameters σ and k

$$\sigma = \omega_0 \sqrt[4]{1+\bar{\alpha}\eta_1^2} = \omega_0 \sqrt[4]{1+2\bar{\alpha}}, \tag{14}$$

$$k = \sqrt{\frac{\bar{\alpha}\eta_1^2}{2(1+\bar{\alpha}\eta_1^2)}} = \sqrt{\frac{\sqrt{1+2\bar{\alpha}}-1}{2\sqrt{1+2\bar{\alpha}}}}, \tag{15}$$

and the following equation for the factor η_1:

$$\eta_1^4 + \frac{2}{\bar{\alpha}}\eta_1^2 - \frac{2}{\bar{\alpha}} = 0. \tag{16}$$

In these relationships,

$$\bar{\alpha} = \alpha\frac{2gH}{\omega_0^4} \tag{17}$$

is the dimensionless parameter of nonlinearity. The eqn (16) yields:

$$\eta_1 = \sqrt{\frac{\sqrt{1+2\bar{\alpha}}-1}{\bar{\alpha}}}. \tag{18}$$

In the case of a hard spring, the maximum acceleration always takes place at the same moment of time as the maximum displacement, and can be determined directly from the eqn (9):

$$\ddot{x}_{max} = -\omega_0^2 x_{max} - \alpha x_{max}^3 = -\eta_2 \omega_0 \sqrt{2gH}. \tag{19}$$

Here the factor

$$\eta_2 = \eta_1\sqrt{1+2\bar{\alpha}} \tag{20}$$

reflects the effect of the nonlinearity on the maximum acceleration.

The product of the maximum displacement and the maximum acceleration is

$$R = x_{max}\ddot{x}_{max} = \eta_p(-2gH), \tag{21}$$

where the factor

$$\eta_p = \frac{1+2\bar{\alpha}-\sqrt{1+2\bar{\alpha}}}{\bar{\alpha}} = \frac{2\sqrt{1+2\bar{\alpha}}}{1+\sqrt{1+2\bar{\alpha}}} \tag{22}$$

accounts for the effect of the nonlinearity on this product.

Soft spring

In the case of a soft spring, the solution to the eqn (9) can be sought in the form:

$$x = \frac{\sqrt{2gH}}{\omega_0}\eta_1 sn(\sigma t+\varepsilon, k) = \frac{\sqrt{2gH}}{\omega_0}\eta_1 snu, \tag{23}$$

where the notation is the same as in the previous section. By differentiation, we find:

$$\dot{x} = \frac{\sqrt{2gH}}{\omega_0}\eta_1\sigma cnu\, dnu, \tag{24}$$

$$\ddot{x} = -\frac{\sqrt{2gH}}{\omega_0}\eta_1\sigma^2 snu(1+k^2-2k^2sn^2u). \tag{25}$$

Substituting (23) and (25) into the eqn (9), and considering the initial conditions (10), we obtain the following formulas for the parameters σ and k:

$$\sigma = \omega_0\sqrt{1-\bar{\alpha}\eta_1^2} = \omega_0\sqrt{1-2\bar{\alpha}}, \tag{26}$$

$$k = \sqrt{\frac{\bar{\alpha}\eta_1^2}{2(\bar{\alpha}\eta_1^2-1)}} = \sqrt{\frac{1-\sqrt{1-2\bar{\alpha}}}{2\sqrt{1-2\bar{\alpha}}}}, \tag{27}$$

and the following equation for the factor η_1:

$$\eta_1^4 - \frac{2}{\bar{\alpha}}\eta_1^2 + \frac{2}{\bar{\alpha}} = 0. \tag{28}$$

This equation yields:

$$\eta_1 = \sqrt{\frac{1-\sqrt{1-2\bar{\alpha}}}{\bar{\alpha}}}. \tag{29}$$

In order to determine the maximum acceleration we differentiate the eqn (9) with respect to time t

$$\dddot{x} = (-\omega_0^2+3\alpha x^2)\dot{x}. \tag{30}$$

and equate the obtained expression to zero. The condition $\dddot{x} = 0$ indicates that the (absolute) maxima of the acceleration (deceleration) $\ddot{x}$ occur in two cases: (1) when $\dot{x} = 0$ and (2) when

$$-\omega_0^2+3\alpha x^2 = 0, \tag{31}$$

and the displacement is equal to

$$x = x_* = \frac{\omega_0}{\sqrt{3\alpha}} = \frac{1}{\sqrt{3\bar{\alpha}}}\frac{\sqrt{2gH}}{\omega_0}. \tag{32}$$

This value can be substantially smaller than the maximum displacement x_{max}.

In the first case, the maximum acceleration occurs simultaneously with the maximum displacement and can be evaluated as

$$\ddot{x}_{max} = -\omega_0^2 x_{max}+\alpha x_{max}^3 = -\eta_2\omega_0\sqrt{2gH}, \tag{33}$$

where the factor

$$\eta_2 = \eta_1\sqrt{1-2\bar{\alpha}} \tag{34}$$

considers the effect of the nonlinearity on the maximum acceleration. The product of the maximum displacement and the maximum acceleration is still given by the eqn (21), in which, however, the factor η_p, reflecting the effect of the nonlinearity, is expressed as:

$$\eta_p = \eta_1\eta_2 = \frac{\sqrt{1-2\bar{\alpha}}-(1-2\bar{\alpha})}{\bar{\alpha}} = \frac{2\sqrt{1-2\bar{\alpha}}}{1+\sqrt{1-2\bar{\alpha}}}. \tag{35}$$

As evident from the formulae (29), (34), and (35), the motions cease to be vibrational, when the parameter $\bar{\alpha}$ exceeds 1/2.

In the second case, i.e. in the case when the condition (31) is fulfilled, the maximum acceleration can be evaluated by the formula:

$$\ddot{x} = \ddot{x}_* = -\omega_0^2 x_*+\alpha x_*^3 = -\frac{2}{3}\frac{\omega_0^3}{\sqrt{3\alpha}} = -\frac{2\omega_0\sqrt{2gH}}{3\sqrt{3\bar{\alpha}}}. \tag{36}$$

The ratio of the maximum displacement x_{max} (occurring at the end of the "breaking time"), to the displacement x_* (occurring at the moment of time when the acceleration reaches its maximum) is

$$\frac{x_{max}}{x_*} = \sqrt{3(1-\sqrt{1-2\bar{\alpha}})}. \tag{37}$$

This ratio is equal to unity when the parameter $\bar{\alpha}$ of nonlinearity is equal to $\bar{\alpha} = 5/18 = 0.2778$. This means that for a system characterized by a dimensionless parameter $\bar{\alpha}$ of nonlinearity not exceeding this value, the maximum acceleration occurs at the moment of time when $x = x_{max}$ (i.e. when the velocity $\dot{x}$ is zero), and can be evaluated either by the formula (33) or by the formula (36). For systems characterized by $\bar{\alpha}$ values exceeding $\bar{\alpha} = 0.2778$, the maximum displacement x_{max} is greater than the displacement x_* (which takes place at the moment of time when the maximum acceleration occurs). In the extreme case of $\bar{\alpha} = 0.5$ the ratio (37) becomes as high as $\sqrt{3} = 1.7321$. In a situation when the maximum displacement x_{max} is greater than the value x_* predicted by the formula (32), the maximum acceleration occurs at the moment of time when the displacement x is equal to x_*. This acceleration can be computed by the formula (36), which for $\bar{\alpha} = 0.2778$, yields:

$$\ddot{x}_* = -0.7307\omega_0\sqrt{2gH}.$$

This value is by a factor of 0.7307 smaller than the maximum acceleration in a linear system. In the extreme case of $\bar{\alpha} = 0.5$ the formula (36) yields:

$$\ddot{x}_* = -0.5443\omega_0\sqrt{2gH},$$

which is by a factor of 0.5443 smaller than the maximum acceleration in a linear system.

The product of the maximum displacement and the maximum acceleration in a system for which $\bar{\alpha} > 0.2778$ is

$$R = x_{max}\ddot{x}_* = \eta_p(-2gH), \tag{37}$$

where the factor

$$\eta_p = \frac{2}{3\bar{\alpha}}\sqrt{\frac{1-\sqrt{1-2\bar{\alpha}}}{3}}, \quad 0.2778 \leqslant \bar{\alpha} \leqslant 0.5, \tag{38}$$

considers the effect of the nonlinearity. It should be pointed out that in the case $\bar{\alpha} > 0.2778$ the maximum acceleration $\ddot{x}_*$ occurs earlier than the maximum displacement x_{max}, so that the maximum displacement x_{max} and the maximum acceleration $\ddot{x}_*$ are computed for different moments of time. As follows from the formula (38), the factor η_p changes from $\eta_p = 0.8$, when $\bar{\alpha} = 0.2778$, to $\eta_p = 0.7698$, when $\bar{\alpha} = 0.5$. The analysis of this formula indicates that the factor η_p has its minimum value $\eta_p = 1/\sqrt{2} = 0.7071$, when the parameter $\bar{\alpha}$ is equal to $\bar{\alpha} = \frac{4}{9} = 0.4444$.

The results of the performed analysis indicate that the formulae (18), (20), and (22), obtained for the case of a spring with a hard characteristic, are applicable in the case of a spring with a soft characteristic as well, if the values of the parameter of nonlinearity $\bar{\alpha}$ are simply extended into the area of negative $\bar{\alpha}$ values. The calculated factors η_1, η_2, and η_p, which reflect the effect of the nonlinearity on the maximum displacement, maximum acceleration, and their product, respectively, are shown for different $\bar{\alpha}$ values in Table 1 and plotted in Fig. 1.

DISCUSSION

As evident from the calculated data, a spring with a hard characteristic results in appreciably lower displacements and significantly higher accelerations than a linear spring. The cumulative effect of the application of a hard spring is such, that the adverse effect of the nonlinearity on the maximum acceleration exceeds its favorable effect on the maximum

Table 1.

$\bar{\alpha}$	-0.5	-0.4444	-0.4	-0.3	-0.2778	-0.2	-0.1	0	0.1	0.5	1.0	2.0	5.0	∞
η_1	$\sqrt{2}$	1.2247	1.1756	1.1069	1.0955	1.0616	1.0275	1	0.9770	0.9102	0.8556	0.7862	0.6807	0
η_2	0.5443	0.5774	0.6086	0.7027	0.7307	0.8223	0.9190	1	1.0702	1.2872	1.4819	1.7580	2.2576	∞
η_p	0.7698	0.7071	0.7154	0.77779	0.8000	0.8730	0.9443	1	1.0456	1.1716	1.2680	1.3821	1.5368	2

$$\eta_1 = \sqrt{\frac{\sqrt{1+2\bar{\alpha}}-1}{\bar{\alpha}}}, \quad \eta_2 = \begin{cases} \eta_1\sqrt{1-2\bar{\alpha}}, & \text{for } \bar{\alpha} \geqslant -\frac{5}{18} = -0.2778 \\ \dfrac{2}{3\sqrt{-3\bar{\alpha}}}, & \text{for } 0.2778 \geqslant \bar{\alpha} \geqslant -0.5 \end{cases} \quad \eta_p = \eta_1\eta_2$$

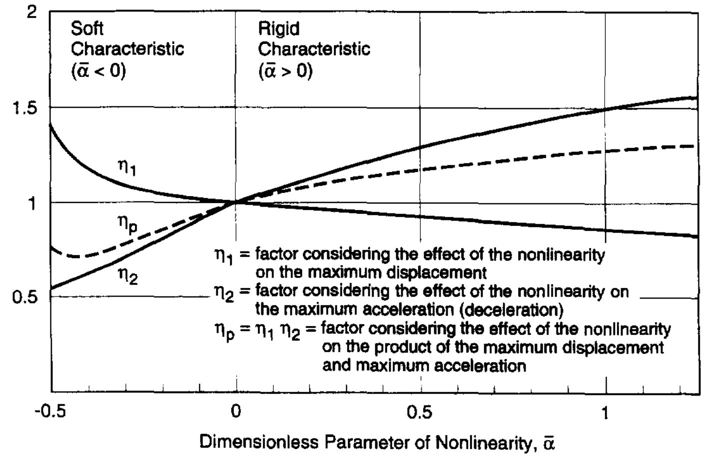

Fig. 1. Factors considering the effect of the nonlinearity on the maximum displacement (η_1), maximum acceleration (η_2), and the product of the maximum displacement and maximum acceleration as functions of the dimensionless parameter of nonlinearity, $\bar{\alpha}$.

displacement, and therefore the factor η_p which considers the effect of the nonlinearity on their product increases with an increase in the degree of the nonlinearity. In the case of a spring with a soft characteristic, the maximum displacement increases and the maximum acceleration decreases with an increase in the degree of nonlinearity. The cumulative effect of the application of a soft spring is such that the factor $\eta_p$decreases with the increase in the degree of nonlinearity, when the parameter $\bar{\alpha}$ changes from zero to 0.4444, and increases with the further increase in the nonlinearity. The minimum value of the factor η_p is equal to $\sqrt{2} = 0.7071$ and takes place for $\bar{\alpha} = 0.4444$.

Despite the adverse effect of a hard spring on the maximum acceleration, application of such a spring might still be feasible, if the structural element is able to withstand high accelerations and there is a strong need to reduce the "stopping distance" by any means. As far as the spring with a soft characteristic is concerned, its application might be advisable when the structural element to be protected is unable to withstand high accelerations, and, at the same time, the restriction on the maximum displacement is not very stringent. As to the product of the maximum displacement and the maximum acceleration, application of a spring with a soft characteristic can result in an appreciable reduction in this characteristic in comparison with a linear case.

It should be emphasized, however, that by no means the $\bar{\alpha}$ value shall be permitted to exceed 1/2, otherwise an extremely undesirable "rigid impact" is possible. In order to assess the degree of the acceptable negative nonlinearity, one can use the eqn (17), and write the condition $\bar{\alpha} > \frac{1}{2}$ as follows:

$$H < \frac{\omega_0^4}{g} \frac{1}{4\alpha}, \tag{39}$$

or

$$\alpha < \frac{\omega_0^4}{4gH}. \tag{40}$$

The condition (39) indicates that for a soft nonlinear spring, characterized by the linear frequency of vibrations ω_0 and the parameter of nonlinearity α, the maximum drop height should not be allowed to exceed the value

$$H = \frac{\omega_0^4}{g}\frac{1}{4\alpha}. \tag{41}$$

The condition (40) indicates that a soft protective spring should be designed in such a way that the parameter α of nonlinearity does not exceed the value

$$\alpha = \frac{\omega_0^4}{4gH}. \tag{42}$$

As has been found in the previous section, the mechanical behavior of a spring protector with a soft characteristic depends on whether the zero velocity is reached below or above the maximum value of the restoring force. This value is characterized by the parameter $\bar{\alpha} = \frac{5}{18} = 0.2778$. If the nonlinear restoring force and the drop height are such that the actual parameter $\bar{\alpha}$ of nonlinearity is below this value, the maximum displacement and the maximum acceleration occur at the same moment of time (which is the moment of time when the velocity is zero). If the value of the parameter $\bar{\alpha}$ exceeds 0.2778, then the maximum acceleration occurs prior to the maximum displacement. In such a case the maximum displacement takes place when the velocity of the system is zero, while the maximum acceleration occurs when the velocity is still finite.

PROBABILISTIC APPROACH

Since the actual drop height H is never known with certainty, it should be treated as a random variable. Let us assume, for instance, that this height is distributed in accordance with the Rayleigh law:

$$f_H(H) = \frac{H}{H_p^2}\exp\left(-\frac{H^2}{H_p^2}\right). \tag{43}$$

Here H is the random drop height and H_p is its most likely value. The probability that the drop height H exceeds a certain level H_* is

$$P = P(H > H_*) = \int_{H_*}^{\infty} f_H(H)\,\mathrm{d}H = \mathrm{e}^{-H_*^2/2H_p^2}.$$

Solving this equation for H_*, we have:

$$H_* = H_p\sqrt{-2\ln P}. \tag{44}$$

Using the formula (42), we conclude that the spring characteristic of a protective soft spring should be chosen in such a way that the parameter α of nonlinearity does not exceed the value

$$\alpha = \frac{\omega_0^4}{4gH_p\sqrt{-2\ln P}}, \tag{45}$$

and the probability P should be chosen sufficiently low. This formula can be written also, considering (17), as

$$\bar{\alpha} = \frac{1}{2\sqrt{-2\ln P}}. \tag{46}$$

As an illustration, consider the example examined in Suhir and Burke (1994). Let the

Table 2.

K_1 kgf/mm	50	100	150	200	250	300
ω_0, sec^{-1}	1650	2307	2792	3187	3520	3810
α, 1/m^2 sec^2	5.0631×10^{10}	19.350×10^{10}	41.571×10^{10}	70.474×10^{10}	104.870×10^{10}	143.941×10^{10}

most likely drop height be $H_p = 5$ ft $= 1.524$ m, and the probability P of a rigid impact be chosen as low as $P = 0.05$. Then, the formula (45) yields: $\alpha = 0.006831\ \omega_0^4$. The calculated values of the parameter α for different ω_0 values (γ_1 values in Suhir and Burke (1994)) are shown in Table 2. The formula (46), for $P = 0.05$, yields: $\bar{\alpha} = 0.2043$. Then, as follows from the Table 1 and Fig. 1, the application of a spring with a soft cubic characteristic results in the increase in the maximum displacement by about 6%, in the decrease in the maximum acceleration by about 18%, and in the decrease in their product by about 13%.

DESIGN CONSIDERATIONS: THE CASE OF A SOFT SPRING

The design considerations in the case of a soft spring are not as straightforward as in the case of a spring with a hard characteristic.

Let the structural element protected by a soft spring, which is characterized by the degree of nonlinearity of $\bar{\alpha} \leqslant 0.2778$, be able to withstand an acceleration of the magnitude $\ddot{x}_{max}$, while the maximum allowable displacement ("stopping distance") is x_{max}. The objective of the analysis which follows is to determine the required restoring force, i.e. the linear frequency ω_0 and the parameter of nonlinearity α.

The eqn (9) can be written, in the case of a soft spring, as follows:

$$\frac{d}{dt}\left(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega_0^2 x^2 - \tfrac{1}{4}\alpha x^4\right) = 0,$$

or

$$\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega_0^2 x^2 - \tfrac{1}{4}\alpha x^4 = C,$$

where C is the constant of integration. Since $\dot{x} = \sqrt{2gH}$, when $x = 0$, we obtain: $C = gH$, and the phase diagram of the system in question is described by the equation

$$\dot{x}^2 = 2gH - \omega_0^2 x^2 + \tfrac{1}{2}\alpha x^4. \tag{47}$$

Since $x = x_{max}$, when $\dot{x} = 0$, we obtain:

$$\omega_0^2 x_{max}^2 - \tfrac{1}{2}\alpha x_{max}^4 = 2gH. \tag{48}$$

It is noteworthy that by using the notation $x_{max} = \eta_1(\sqrt{2gH}/\omega_0)$, this equation leads to the eqn (28), which was obtained on the basis of a different approach, namely, of an approach using the complete solution to the equation of motion.

Solving the eqn (48) along with the equation

$$\omega_0^2 x_{max} - \alpha x_{max}^3 = -\ddot{x}_{max},$$

which can be obtained directly from (9), for the unknowns ω_0 and α, we find:

$$\omega_0^2 = \frac{4gH}{x_{max}^2}\left(1 - \tfrac{1}{2}\eta_p\right), \quad \alpha = \frac{4gH}{x_{max}^4}(1 - \eta_p), \tag{49}$$

where the factor

$$\eta_p = -\frac{\ddot{x}_{max}x_{max}}{2gH}, \quad 0 \leqslant \eta_p \leqslant 1, \tag{50}$$

considers the effect of the nonlinearity on the product $x_{max}\ddot{x}_{max}$. Introducing (49) into (17), we obtain the following relationship between this product and the dimensionless parameter $\bar{\alpha}$ of nonlinearity:

$$\bar{\alpha} = \frac{1-\eta_p}{2-\eta_p}, \quad 0 \leqslant \bar{\alpha} \leqslant \tfrac{1}{2}, \quad 0 \leqslant \eta_0 \leqslant 1. \tag{51}$$

Clearly, the linear spring constant (i.e. the spring constant in this case of small displacements) can be evaluated as

$$K = M\omega_0^2 = \frac{4MgH}{x_{max}^2}(1-\tfrac{1}{2}\eta_p). \tag{52}$$

Thus, in order to evaluate the constant K (and the linear frequency ω_0) and the required parameter α of nonlinearity for the given maximum acceleration, $\ddot{x}_{max}$, the available maximum displacement, x_{max}, and the given drop height, H, one should calculate first the product η_p by the formula (50), and then apply the formulas (49) and (52).

If a soft spring protector is characterized by the parameter of nonlinearity $\bar{\alpha} > 0.2778$, the maximum acceleration should be calculated by the formula (36) which, considering the relationship (48), yields:

$$\omega_0^6 - \frac{27}{2}\frac{\ddot{x}_*^2}{x_{max}^2}\omega_0^2 + 27gH\frac{\ddot{x}_*^2}{x_{max}^4} = 0, \tag{53}$$

or

$$\zeta = \frac{gH}{x_{max}\ddot{x}_*} = \sqrt{\frac{1-\xi}{(\frac{2}{3}\xi)^3}}, \tag{54}$$

where the notation

$$\xi = \omega_0^2\frac{x_{max}^2}{2gH}, \quad 0 \leqslant \xi \leqslant 1, \tag{55}$$

is used. The tabulated values of the parameter $\zeta = gH/x_{max}\ddot{x}_*$ are given in Table 3 and

Table 3.

ζ	0	0.1	0.2	0.4	0.6	0.8	0.9	1
ξ	∞	55.113	18.371	5.6250	2.500	1.1482	0.6804	0

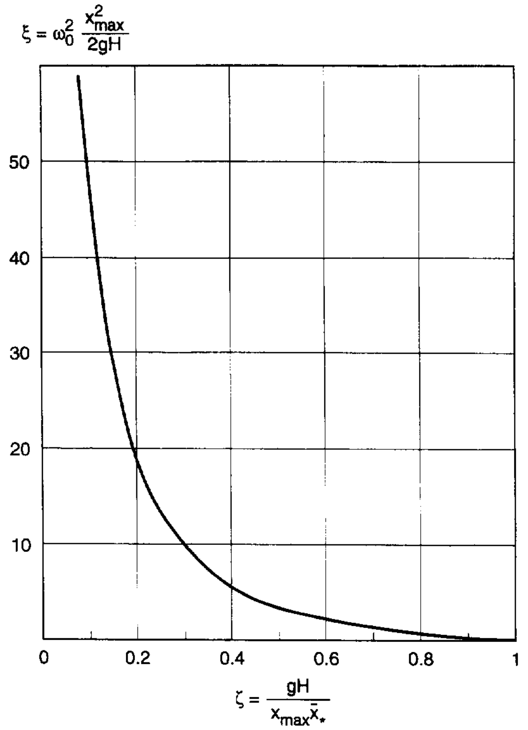

Fig. 2. Factor of the maximum displacement vs factor of the "quality" of the protective spring.

plotted in Fig. 2. After the frequency ω_0 is determined, the α value can be easily calculated as

$$\alpha = \frac{4\omega_0^6}{27\ddot{x}_*^2}. \tag{56}$$

COMPARISON WITH AN IDEAL SHOCK ABSORBER

The requirement for an ideal shock absorber is rather straightforward: if the element to be protected is able to sustain an acceleration of the magnitude a, an ideal absorber should be able to provide this acceleration, for the minimum stopping distance, during the entire time of "breaking". With the constant acceleration a, the displacement $x = x(t)$ of the element can be evaluated by the well-known formula:

$$x = \sqrt{2gH}t - \tfrac{1}{2}at^2. \tag{57}$$

Then the velocity $\dot{x} = \dot{x}(t)$ is

$$\dot{x} = \sqrt{2gH} - at. \tag{58}$$

This expression, with the condition $\dot{x} = 0$, results in the following formula for the "breaking" time:

$$T = \frac{\sqrt{2gH}}{a}. \tag{59}$$

With $t = T$, the eqn (57) yields:

$$x_{max} = \frac{gH}{a}, \tag{60}$$

or

$$a = g\frac{H}{x_{max}}. \tag{61}$$

The formula (61) enables one to evaluate the constant acceleration a for the given drop height H and the required maximum stopping distance x_{max}. Note that in such a case the velocity/displacement relationship of the shock absorbing material is characterized by the equation

$$\dot{x} = \sqrt{2(gH - ax)}. \tag{62}$$

This relationship can be obtained from the eqns (57) and (58) by eliminating the time t. It is doubtful that a material with such a characteristic exists or can be easily developed.

From the formulas (60) or (61) we find that the product

$$R = x_{max}\ddot{x}_{max} = x_{max}a = -gH \tag{63}$$

of the maximum displacement and the maximum acceleration is half of the value in the case of a linear spring.

As is evident from Table 1, a nonlinear spring with a soft characteristic and a sufficiently high level of the parameter $\bar{\alpha}$ of nonlinearity can be quite effective. As has been shown above, the product of the maximum acceleration in a soft spring protector, characterized by the value of the dimensionless parameter $\bar{\alpha} = \frac{4}{9} = 0.444$, is by a factor of $1/\sqrt{2} = 0.7071$ lower than in the case of a linear system (although still by a factor of $\sqrt{2} = 1.41$ larger than in the case of an ideal shock absorber). However, if the drop height H follows indeed the Rayleigh law of the probability distribution, then, using the formula (46), we conclude that the probability that the above value of the parameter $\bar{\alpha}$ is exceeded is as high as $P = 0.755$. Therefore, if the maximum drop height is not known with certainty and obeys the Rayleigh law of the probability distribution, the advantages of a soft spring cannot be utilized to the full extent because of the high probability of a rigid impact. It could very well be, however, that the actual law of the probability distribution for the drop height is different and, hopefully, more favorable than the rather conservative Rayleigh law used in this analysis.

As an illustration, examine a situation when the drop height is $H = 5$ ft $= 1.524$ m, the weight of the element to be protected is $P = 0.5$ kgf, and the allowable acceleration is $a = 2000$ g. If a linear spring is used, the frequency of vibrations is $\omega_0 = \ddot{x}_{max}/\sqrt{2gH} = 3588$ 1/sec, and the spring constant is $K = M\omega_0^2 = 656154$ kgf/m. The maximum stopping distance with such a spring is $x_{max} = \sqrt{2gH}/\omega_0 = 1.52$ mm. In the case of an ideal viscous shock absorber, the stopping distance is only half of this value: $x_{max} = gH/a = 0.762$ mm. In the case of a nonlinear spring with a soft cubic characteristic, one can choose, using a conservative approach, $\bar{\alpha} = 0.2043$ (which corresponds to the probability $P = 0.05$ that the value $\bar{\alpha} = 0.5$ is exceeded). Then, solving the eqn (51) for the η_p value, we obtain: $\eta_p = (1-2\bar{\alpha})/(1-\bar{\alpha}) = 0.7432$, and the formula (50) yields: $x_{max} = 1.133$ mm. This value is by a factor of 0.743 smaller than the maximum displacement in the case of a linear spring, although by a factor of 1.487 greater than the maximum displacement in the case of an ideal viscous shock absorber.

EFFECT OF VISCOUS DAMPING

In the previous section we compared a soft nonlinear spring with an ideal viscous shock absorber. Strictly speaking, we used a "double standard" when making such a comparison: the capabilities of a nonlinear undamped system were evaluated against the

Table 4.

η	0	0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0
η_1	1	0.8226	0.7561	0.6715	0.6029	0.5463	0.4988	0.4240	0.3679
η_2	1	0.8801	0.8209	0.8134	0.8635	1.0	1.2	1.6	2.0
$\eta_p = \eta_1\eta_2$	1	0.7592	0.6207	0.5462	0.5206	0.5463	0.5986	0.6784	0.7358

capabilities of a highly damped and, in effect, also nonlinear system. Let us assess whether viscous damping can have a significant effect on the dynamic response. In the analysis carried out for a linear system (Suhir (1995)) the formulas for the maximum displacement and the maximum acceleration were obtained as a result of solving the equation

$$\ddot{x}+2\eta\omega_0\dot{x}+\omega_0^2x = 0, \quad x(0) = 0, \quad \dot{x}(0) = \sqrt{2gH}$$

as follows:

$$x_{max} = \eta_1\frac{\sqrt{2gH}}{\omega_0}, \quad \ddot{x}_{max} = -\eta_2\omega_0\sqrt{2gH}. \tag{64}$$

Here the factors η_1 and η_2, considering the effect of viscous damping on the maximum displacement and the maximum acceleration, can be evaluated by the formulae:

$$\eta_1 = \begin{cases} \exp\left[-\eta\dfrac{\arcsin\sqrt{1-\eta^2}}{\sqrt{1-\eta^2}}\right], & 0 \leqslant \eta \leqslant 1, \\ (\eta+\sqrt{\eta^2-1})^{-\eta/\sqrt{\eta^2-1}}, & \eta \geqslant 1 \end{cases} \tag{65}$$

$$\eta_2 = \begin{cases} \exp\left[-\eta\dfrac{\arcsin[(1-4\eta^2)\sqrt{1-\eta^2}]}{\sqrt{1-\eta^2}}\right], & 0 \leqslant \eta \leqslant 0.5, \\ 2\eta, & \eta \geqslant 0.5 \end{cases} \tag{66}$$

where

$$\eta = \frac{r}{\omega_0} = \frac{R}{2\sqrt{KM}}$$

is the damping ratio, and R is the damping coefficient. The calculated factors η_1, η_2, and their product $\eta_p = \eta_1\eta_2$ are shown in Table 4 and plotted in Fig. 3. As evident from the calculated data, moderate damping can bring down substantially the product η_p of the maximum displacement and the maximum acceleration. The minimum η_p value takes place for $\eta = 0.4$ and is only $\eta_p = 0.5206$. Although a detailed analysis of the effect of viscous damping on the response of a nonlinear system is beyond the scope of this study, there is a reason to believe that damping can improve appreciably the dynamic response of a structural element protected by a spring with a soft characteristic, perhaps, making it even competitive with an ideal absorber.

FORMALIZATION OF THE OBTAINED RESULTS ON THE BASIS OF THE PHASE DIAGRAM ANALYSIS

In the previous analyses, it has been shown that the maximum displacements and the maximum accelerations can be obtained either on the basis of the complete solutions to the governing equations of motion, or directly from the equation

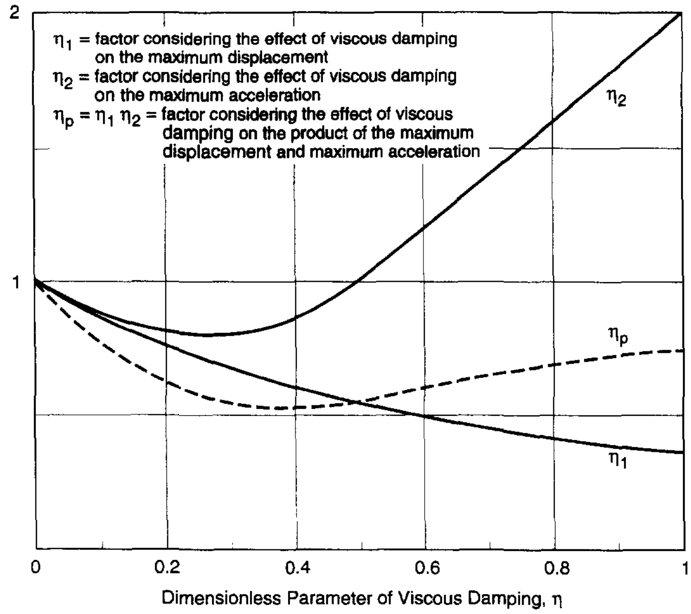

Fig. 3. Factors considering the effect of viscous damping on the maximum displacement (η_1), maximum acceleration (η_2), and the product of the maximum displacement and maximum acceleration as functions of the dimensionless parameter of viscous damping, η.

$$\dot{x}^2 = 2gH - \omega_0^2 x^2 - \tfrac{1}{2}\alpha x^4$$

for the phase diagram. Whatever the approach, we treated the cases of a hard and a soft spring separately to make the analyses physically clear and meaningful, and to avoid confusion. In the treatment which follows we formalize the obtained results using the phase diagram approach which readily permits one to interpret these results for either hard or soft nonlinearity, without "switching" the definition of the parameter of nonlinearity from positive to negative.

Introducing dimensionless displacement $\bar{x} = \omega_0/\sqrt{2gH}$ and dimensionless time $\bar{t} = \omega_0 t$, the above phase diagram equation can be written as

$$\frac{d\bar{x}}{d\bar{t}} = \sqrt{1 - \bar{x}^2 - \tfrac{1}{2}\bar{\alpha}\bar{x}^4}, \tag{67}$$

where the dimensionless parameter of nonlinearity $\bar{\alpha}$ is expressed by the formula (17). Clearly, the dimensionless acceleration is

$$\frac{d^2\bar{x}}{d\bar{t}^2} = -\bar{x} - \bar{\alpha}\bar{x}^3. \tag{68}$$

The $\bar{\alpha}$ value in the eqns (67) and (68) can be either positive (hard spring) or negative (soft spring). The maximum dimensionless displacement $\eta_1 = \bar{x}_{max}$ takes place when the velocity $d\bar{x}/d\bar{t}$ is zero. This leads to the eqn (16) and to the formula (18). As evident from this formula, the vibrational character of the motion ceases to exist for $\bar{\alpha} \leqslant -0.5$. This formula indicates also that the factor η_1 (the maximum dimensionless displacement $\bar{x}_{max}$) changes from $\sqrt{2}$ to zero, when the parameter of nonlinearity changes from -0.5 to infinity.

The dimensionless acceleration (deceleration), a, for the moment of time when the displacement $\bar{x}$ reaches its maximum value η_1 can be obtained from (68) and (18) as

$$a = -\eta_1 - \bar{\alpha}\eta_1^3 = -\eta_1\sqrt{1+2\bar{\alpha}} = -\sqrt{\frac{2(1+2\bar{\alpha})}{\sqrt{1+2\bar{\alpha}}+1}}. \tag{69}$$

As to the maximum dimensionless acceleration (deceleration), a_{max}, it is equal, as evident from (68), to

$$a_{max} = -\frac{2}{3\sqrt{-3\bar{\alpha}}}, \tag{70}$$

and occurs when the displacement $\bar{x}$ is equal to

$$\bar{x}_* = \frac{1}{\sqrt{-3\bar{\alpha}}}. \tag{71}$$

Equating the expressions (69) and (70), and solving the obtained equation for the $\bar{\alpha}$ value, we find: $\bar{\alpha} = -\frac{5}{18} = -0.2778$. Hence, the maximum dimensionless acceleration (deceleration) $\eta_2 = a_{max}$ occurs simultaneously with the maximum displacement in the systems for which the parameter of nonlinearity $\bar{\alpha}$ is larger than -0.2778, otherwise the maximum dimensionless acceleration η_2 occurs earlier than the maximum dimensionless displacement η_1. Thus,

$$\eta_2 = \begin{cases} \dfrac{\sqrt{2(1+2\bar{\alpha})}}{\sqrt{1+2\bar{\alpha}+1}}, & \text{for} \quad -0.2778 \leqslant \bar{\alpha} < \infty \\ \dfrac{2}{3\sqrt{-3\bar{\alpha}}}, & \text{for} \quad -0.5 \leqslant \alpha \leqslant -0.2778 \end{cases} \tag{72}$$

The dimensionless velocity $\mathrm{d}\bar{x}/\mathrm{d}\bar{t}$ at the moment of time when the dimensionless displacement $\bar{x}$ is equal to the value $\bar{x}_*$ is as follows:

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}\bar{t}} = \sqrt{1+\frac{5}{18\bar{\alpha}}}. \tag{73}$$

For $\bar{\alpha} = -0.2778$ the maximum values of the dimensionless displacement and acceleration still take place simultaneously and are $\eta_1 = \sqrt{\frac{6}{5}} = 1.0954$ and $\eta_2 = \frac{2}{3}\sqrt{\frac{6}{5}} = 0.7303$, respectively. In the extreme case, $\bar{\alpha} = -0.5$, the dimensionless displacement $\bar{x}_* = \sqrt{\frac{2}{3}} = 0.8165$, corresponding to the maximum dimensionless acceleration $\eta_2 = \frac{2}{3}\sqrt{\frac{2}{3}} = 0.5443$ is substantially smaller than the maximum dimensionless displacement $\eta_1 = \sqrt{2} = 1.4142$, while the velocity $\mathrm{d}\bar{x}/\mathrm{d}\bar{t}$ at this moment of time, is as high as $\mathrm{d}\bar{x}/\mathrm{d}\bar{t} = \frac{2}{3} = 0.6667$, and is closer to the initial velocity (which is equal to unity), than to zero (which corresponds to the maximum displacement).

The product $\eta_p = \eta_1\eta_2$ can be calculated as

$$\eta_p = \begin{cases} \dfrac{2\sqrt{1+2\bar{\alpha}}}{1+\sqrt{1+2\bar{\alpha}}}, & \text{for} \quad -0.2778 \leqslant \bar{\alpha} \leqslant \infty \\ \dfrac{2}{3\bar{\alpha}}\sqrt{\dfrac{1-\sqrt{1+2\bar{\alpha}}}{3}}, & \text{for} \quad -0.5 \leqslant \bar{\alpha} \leqslant -0.2778 \end{cases} \tag{74}$$

This product is equal to $\eta_p = 2$ for very large $\bar{\alpha}$, is equal to $\eta_p = \frac{4}{5} = 0.8$ for $\bar{\alpha} = -0.2778$, and is equal to $\eta_p = 4/3\sqrt{3} = 0.7698$ for $\bar{\alpha} = -0.5$. The analysis of the second formula in (74) indicates that the factor η_p reaches its minimum value $\eta_p = 1/\sqrt{2} = 0.7071$ for $\bar{\alpha} = -\frac{4}{9} = -0.4444$.

CONCLUSIONS

The following major conclusions can be drawn from the performed analysis:

—Application of a nonlinear spring with a hard characteristic results in an appreciable decrease in the maximum displacement and in a considerable increase in the maximum acceleration (deceleration), compared to the linear case. The cumulative effect of a hard spring is such that the product of the maximum displacement and the maximum acceleration (which can be considered as a suitable criterion of the efficiency of the shock protector) increases as well. Therefore employment of such a spring can be feasible for elements which are able to withstand high accelerations, while the "stopping distance" has to be made small by any means.

—Application of a nonlinear spring with a soft characteristic results in an appreciable increase in the maximum displacement and a substantial decrease in the maximum acceleration (deceleration), in comparison with the linear case. The cumulative effect of a soft spring is such that the product of the maximum displacement and the maximum acceleration decrease with an increase in the degree of nonlinearity. Therefore the use of such a spring can be recommended for structural elements which cannot withstand high accelerations (decelerations), while the requirement for a short stopping distance is not very critical.

—If the maximum drop height is not known with certainty, the advantages of a soft spring cannot be utilized to a full extent because of the possibility of a "rigid impact". In such a case a probabilistic approach can be effectively used to design a soft spring with a sufficiently low probability of such an impact. In this connection, it should be emphasized that future work should include evaluation of the probability distribution for the actual drop height. It could happen that this function could differ appreciably from, and, perhaps, be more favorable than, the Rayleigh law.

—It is believed that the trend found in this analysis hold, at least qualitatively, for a broad class of soft and hard nonlinear springs, not necessarily with cubic restoring forces.

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